

# Geometric Measure of Pairwise Quantum Discord for Superpositions of Multipartite Generalized Coherent States

M. Daoud<sup>a1</sup> and R. Ahl Laamara<sup>b,c 2</sup>

<sup>a</sup>*Department of Physics, Faculty of Sciences, University Ibnou Zohr,  
Agadir, Morocco*

<sup>b</sup>*LPHE-Modeling and Simulation, Faculty of Sciences, University Mohammed V,  
Rabat, Morocco*

<sup>c</sup>*Centre of Physics and Mathematics, CPM, CNESTEN,  
Rabat, Morocco*

## Abstract

We give the explicit expressions of the pairwise quantum correlations present in superpositions of multipartite coherent states. A special attention is devoted to the evaluation of the geometric quantum discord. The dynamics of quantum correlations under a dephasing channel is analyzed. A comparison of geometric measure of quantum discord with that of concurrence shows that quantum discord in multipartite coherent states is more resilient to dissipative environments than is quantum entanglement. To illustrate our results, we consider some special superpositions of Weyl-Heisenberg,  $SU(2)$  and  $SU(1,1)$  coherent states which interpolate between Werner and Greenberger-Horne-Zeilinger states.

---

<sup>1</sup> email: m\_daoud@hotmail.com

<sup>2</sup> email: ahllaamara@gmail.com

# 1 Introduction

The total correlation in quantum states of a composite system can be split into a classical and a quantum parts. Entanglement is a kind of quantum correlation without classical counterpart. It has triggered off many efforts for a deeper understanding of the difference between classical and quantum correlations. Nowadays, it is well established that entanglement is not only a concept in quantum physics but also a fundamental resource for quantum information processing and necessary for some quantum information tasks like quantum teleportation, quantum cryptography and universal quantum computing (for review, see [1, 2, 3, 4]). However, some recent investigations showed that entanglement is only a one special kind of quantum correlations. Indeed, unentangled quantum states can also possess quantum correlations which play a relevant role in improving quantum communication and information protocols better than their classical counterparts [5, 6]. Therefore, as the non-classicality of correlations present in bipartite and multipartite quantum states is not due solely to the presence of entanglement, there was a need of a measure to characterize and quantify the non-classicality or quantumness of correlations which goes beyond entanglement. A recent catalogue of measures for non classical correlations is presented in [7]. The most popular among them is the so-called quantum discord introduced by Henderson and Vedral [8] and (independently) Ollivier and Zurek [9] who showed that when entanglement is subtracted from total quantum correlation, there remain correlations that are not entirely classical of origin. Now, it is commonly accepted that the most promising candidate to measure quantum correlations is quantum discord. It has attracted considerable attention and continues to be intensively investigated in many contexts [7]. The quantum discord is defined as the difference between quantum mutual information and classical correlation in a bipartite system [8, 9]. It has been calculated explicitly only for a rather limited set of two-qubit quantum states and expressions for more general quantum states are still not known. This is essentially due to the fact that the evaluation of quantum discord involves an optimization procedure which is in general a difficult task to perform. To overcome this difficulty a geometric measure for quantum discord was proposed recently [10]. This is defined, by means of Hilbert-Schmidt norm, as the nearest distance between the quantum state under consideration and the zero-quantum discord states. The explicit expressions of the geometric quantum discord have been obtained only in some few cases including Gaussian states [11, 12] and superpositions of Dicke states [13].

In this paper we shall be mainly concerned with the geometric measure of quantum discord of generalized coherent states. The main reason is that the coherent states constitute a special instance of non orthogonal states whose entanglement properties have received a special attention during the last years (for a recent review see [14] and references therein). In this respect, it is important to investigate the quantum correlations in such quantum states beyond entanglement. Also, the coherent states are of paramount importance in physics (e.g., in quantum optics and quantum information theory) and mathematical physics (e.g., in probability theory, applied group theory, path integral formalism and

theory of analytic functions) [15, 16, 17, 18].

The paper is organized as follows. In order to study the pairwise quantum correlations, we present in Section 2 two different schemes of bipartite partitioning and qubit mapping of a balanced superposition of multipartite coherent states. Section 3 is devoted to the explicit derivation of the geometric measure of quantum discord. The dynamical evolution of bipartite quantum correlations (entanglement and quantum discord) under a dephasing channel is considered in Section 4. Finally, as illustration, some special cases are considered in the last section. Concluding remarks close this paper.

## 2 Superpositions of multipartite coherent states, bipartite partitioning and qubit mapping

### 2.1 Superpositions of multipartite coherent states

We begin by recalling some elements of the Perelomov group theoretic procedure to construct coherent states for a quantum system whose dynamical symmetry is described by a Lie group  $G$  (connected and simply connected, with finite dimension). Let  $T$  be a unitary irreducible representation of  $G$  acting on the Hilbert space of the system. The construction of Perelomov coherent states requires a specific choice of the reference (the ground) state. A coherent state  $|\Omega\rangle$  is determined by a point  $\Omega$  in the coset space  $G/H$  where the isotropy subgroup  $H \subset G$  consists of all the group elements that leave the reference state invariant. The dimension of the coset space  $G/H$  determines the number of the complex variables labeling the Perelomov coherent states. A very important property is the identity resolution in terms of the coherent states:

$$\int_{G/H} d\mu(\Omega) |\Omega\rangle\langle\Omega| = I \quad (1)$$

where  $d\mu(\Omega)$  is the invariant integration measure on  $G/H$  and the integration is over the whole manifold  $G/H$  and  $I$  is the identity operator on the Hilbert space. The choice of the reference state leads to systems consisting of states with properties closest to those of classical states [16]. As special examples one may quote the Glauber coherent states associated with the Weyl-Heisenberg group  $H_3$  which are defined on the complex plane  $\mathbb{C} = H_3/U(1)$ , the spin coherent states defined on the unit sphere  $\mathbb{CP}^1 = SU(2)/U(1)$  and  $SU(1,1)$  coherent states defined on unit disc  $\mathbb{CB}^1 = SU(1,1)/U(1)$ . All these coherent states are labeled by a single complex variable.

For a composite system of  $n$  noninteracting quantum subsystems, the Hilbert space is the tensor product of  $n$  copies of single particle Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n.$$

The coherent states are not orthogonal to each other with respect to the positive measure in (1). Thus, the over-completion relation makes possible the expansion of an arbitrary state of the Hilbert

space  $\mathcal{H}$  in terms of coherent states of the quantum system under consideration. It follows that when one has a collection  $n$  of particles or modes, the whole Hilbert space is a tensorial product and any multipartite state can be written as a superposition of tensorial product of the coherent states  $|\Omega_1\rangle \otimes |\Omega_2\rangle \cdots \otimes |\Omega_n\rangle \equiv |\Omega_1, \Omega_2, \cdots \Omega_n\rangle$ . Indeed, using the resolution to identity, any state  $|\psi\rangle$  in  $\mathcal{H}$  can be expanded as

$$|\psi\rangle = \int d\mu(\Omega''_1) d\mu(\Omega''_2) \cdots d\mu(\Omega''_n) |\Omega''_1, \Omega''_2, \cdots \Omega''_n\rangle \langle \Omega''_1, \Omega''_2, \cdots \Omega''_n | \psi \rangle \quad (2)$$

reflecting that any multipartite state can be viewed as a superposition of product coherent states. The multipartite state (2) can be reduced to a sum if the function

$$\psi(\Omega''_1, \Omega''_2, \cdots \Omega''_n) = \langle \Omega''_1, \Omega''_2, \cdots \Omega''_n | \psi \rangle$$

is expressed as a sum of delta functions. To simplify our purpose, we set

$$\psi(\Omega''_1, \Omega''_2, \cdots \Omega''_n) = \prod_{i=1}^n \delta(\Omega_i - \Omega''_i) + e^{im\pi} \prod_{i=1}^n \delta(\Omega'_i - \Omega''_i).$$

This gives the following equally weighted or balanced superpositions of multipartite coherent states

$$|\psi\rangle \equiv |\Omega, \Omega', m, n\rangle = \mathcal{N}(|\Omega_1\rangle \otimes |\Omega_2\rangle \otimes \cdots \otimes |\Omega_n\rangle + e^{im\pi} |\Omega'_1\rangle \otimes |\Omega'_2\rangle \otimes \cdots \otimes |\Omega'_n\rangle) \quad (3)$$

where  $m \in \mathbb{Z}$  and  $\mathcal{N}$  is a normalization factor given by

$$\mathcal{N} = [2 + 2p_1 p_2 \cdots p_n \cos m\pi]^{-1/2}$$

where the quantities  $p_i$ , assumed to be reals, stand for the overlapping  $\langle \Omega_i | \Omega'_i \rangle$  between two single particle coherent states. It is important to stress that, from an experimental point of view, superpositions of coherent states are difficult to produce, and fundamentally this could be due to extreme sensitivity to environmental decoherence. Experimental efforts to create superpositions of coherent states was reported in [19]. Note also that evidence of such superpositions first appeared in a study of a certain type of nonlinear Hamiltonian evolution by [20, 21], and the manifestation of superpositions of coherent states was analyzed in detail by [22, 23] (see also [24]).

## 2.2 Pairwise partitioning and qubit mapping

To study the pairwise quantum correlations present in the multipartite coherent state (3), the whole system can be partitioned in two different ways.

### 2.2.1 Pure bipartite states

We first consider the splitting of the entire system into two subsystems; one subsystem containing any  $k$  ( $1 \leq k \leq n-1$ ) particles and the other containing the remaining  $n-k$  particles. In this scheme, one writes the state (3) as

$$|\Omega, \Omega', m, n\rangle = \mathcal{N}(|\Omega\rangle_k \otimes |\Omega\rangle_{n-k} + e^{im\pi} |\Omega'\rangle_k \otimes |\Omega'\rangle_{n-k}) \quad (4)$$

where

$$\begin{aligned} |\Omega\rangle_k &= |\Omega_1\rangle \otimes |\Omega_2\rangle \otimes \cdots \otimes |\Omega_k\rangle & |\Omega'\rangle_k &= |\Omega'_1\rangle \otimes |\Omega'_2\rangle \otimes \cdots \otimes |\Omega'_k\rangle \\ |\Omega\rangle_{n-k} &= |\Omega_{k+1}\rangle \otimes |\Omega_{k+2}\rangle \otimes \cdots \otimes |\Omega_n\rangle & |\Omega'\rangle_{n-k} &= |\Omega'_{k+1}\rangle \otimes |\Omega'_{k+2}\rangle \otimes \cdots \otimes |\Omega'_n\rangle \end{aligned}$$

The whole system can be mapped into a pair of two logical qubits. This can be done by introducing the orthogonal basis  $\{|\mathbf{0}\rangle_k, |\mathbf{1}\rangle_k\}$  defined as

$$|\mathbf{0}\rangle_k = \frac{|\Omega\rangle_k + |\Omega'\rangle_k}{\sqrt{2(1 + p_1 p_2 \cdots p_k)}} \quad |\mathbf{1}\rangle_k = \frac{|\Omega\rangle_k - |\Omega'\rangle_k}{\sqrt{2(1 - p_1 p_2 \cdots p_k)}} \quad (5)$$

for the first subsystem. Similarly, we introduce for the second subsystem, containing the remaining  $n - k$  particles, the orthogonal basis  $\{|\mathbf{0}\rangle_{n-k}, |\mathbf{1}\rangle_{n-k}\}$  given by

$$|\mathbf{0}\rangle_{n-k} = \frac{|\Omega\rangle_{n-k} + |\Omega'\rangle_{n-k}}{\sqrt{2(1 + p_{k+1} p_{k+2} \cdots p_n)}} \quad |\mathbf{1}\rangle_{n-k} = \frac{|\Omega\rangle_{n-k} - |\Omega'\rangle_{n-k}}{\sqrt{2(1 - p_{k+1} p_{k+2} \cdots p_n)}} \quad (6)$$

Reporting the equations (5) and (6) in (4), one has the explicit form of the pure state  $|\Omega, \Omega', m, n\rangle$  in the basis  $\{|\mathbf{0}\rangle_k \otimes |\mathbf{0}\rangle_{n-k}, |\mathbf{0}\rangle_k \otimes |\mathbf{1}\rangle_{n-k}, |\mathbf{1}\rangle_k \otimes |\mathbf{0}\rangle_{n-k}, |\mathbf{1}\rangle_k \otimes |\mathbf{1}\rangle_{n-k}\}$ . It is given by

$$|\Omega, \Omega', m, n\rangle = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_k \otimes |\beta\rangle_{n-k} \quad (7)$$

where

$$\begin{aligned} C_{0,0} &= \mathcal{N}(1 + e^{im\pi}) a_k a_{n-k}, & C_{0,1} &= \mathcal{N}(1 - e^{im\pi}) a_k b_{n-k} \\ C_{1,0} &= \mathcal{N}(1 - e^{im\pi}) a_{n-k} b_k, & C_{1,1} &= \mathcal{N}(1 + e^{im\pi}) b_k b_{n-k} \end{aligned}$$

with

$$\begin{aligned} a_k &= \sqrt{\frac{1 + p_1 p_2 \cdots p_k}{2}} & b_k &= \sqrt{\frac{1 - p_1 p_2 \cdots p_k}{2}} \\ a_{n-k} &= \sqrt{\frac{1 + p_{k+1} p_{k+2} \cdots p_n}{2}} & b_{n-k} &= \sqrt{\frac{1 - p_{k+1} p_{k+2} \cdots p_n}{2}} \end{aligned}$$

involving the overlapping  $p_i = \langle \Omega_i | \Omega'_i \rangle$ ,  $i = 1, 2, \dots, n$ .

### 2.2.2 Mixed bipartite states

The second partitioning scheme can be realized by considering the bipartite reduced density matrix  $\rho_{ij}$ , which is obtained by tracing out all other systems except subsystems or modes  $i$  and  $j$ . There are  $n(n-1)/2$  different density matrices  $\rho_{ij}$ . It is simply verified that the reduced density matrix describing the subsystems  $i$  and  $j$  is

$$\rho_{ij} = \mathcal{N}^2 (|\Omega_i, \Omega_j\rangle \langle \Omega_i, \Omega_j| + |\Omega'_i, \Omega'_j\rangle \langle \Omega'_i, \Omega'_j| + e^{im\pi} q_{ij} |\Omega'_i, \Omega'_j\rangle \langle \Omega_i, \Omega_j| + e^{-im\pi} q_{ij} |\Omega_i, \Omega_j\rangle \langle \Omega'_i, \Omega'_j|). \quad (8)$$

The quantity  $q_{ij}$  occurring in (8) is defined by

$$q_{ij} = p_1 p_2 \cdots \check{p}_i \cdots \check{p}_j \cdots p_n$$

where the notation  $\check{p}_i$  and  $\check{p}_j$  indicates that  $p_i$  and  $p_j$  must be omitted from the product of the overlapping of coherent states. Here also, one can map the reduced system into a pair of two-qubits. In this sense, we introduce, for the subsystem  $i$ , the orthogonal basis  $\{|\mathbf{0}_i\rangle, |\mathbf{1}_i\rangle\}$  defined such that

$$|\Omega_i\rangle \equiv a_i|\mathbf{0}_i\rangle + b_i|\mathbf{1}_i\rangle \quad |\Omega'_i\rangle \equiv a_i|\mathbf{0}_i\rangle - b_i|\mathbf{1}_i\rangle, \quad (9)$$

where

$$a_i = \sqrt{\frac{1+p_i}{2}} \quad b_i = \sqrt{\frac{1-p_i}{2}}.$$

Similarly for the subsystem  $j$ , we introduce a second two dimensional orthogonal basis as

$$|\Omega_j\rangle \equiv a_j|\mathbf{0}_j\rangle + b_j|\mathbf{1}_j\rangle \quad |\Omega'_j\rangle \equiv a_j|\mathbf{0}_j\rangle - b_j|\mathbf{1}_j\rangle, \quad (10)$$

where

$$a_j = \sqrt{\frac{1+p_j}{2}} \quad b_j = \sqrt{\frac{1-p_j}{2}}.$$

Substituting Eqs. (9) and (10) into Eq. (8), we obtain the mixed density matrix

$$\rho_{ij} = \mathcal{N}^2 \begin{pmatrix} 2a_i^2a_j^2(1+q_{ij}\cos m\pi) & 0 & 0 & 2a_ia_jb_ib_j(1+q_{ij}\cos m\pi) \\ 0 & 2a_i^2b_j^2(1-q_{ij}\cos m\pi) & 2a_ia_jb_ib_j(1-q_{ij}\cos m\pi) & 0 \\ 0 & 2a_ia_jb_ib_j(1-q_{ij}\cos m\pi) & 2a_j^2b_i^2(1-q_{ij}\cos m\pi) & 0 \\ 2a_ia_jb_ib_j(1+q_{ij}\cos m\pi) & 0 & 0 & 2b_i^2b_j^2(1+q_{ij}\cos m\pi) \end{pmatrix} \quad (11)$$

in the basis  $\{|\mathbf{0}_i\mathbf{0}_j\rangle, |\mathbf{0}_i\mathbf{1}_j\rangle, |\mathbf{1}_i\mathbf{0}_j\rangle, |\mathbf{1}_i\mathbf{1}_j\rangle\}$ .

### 3 Quantum discord

#### 3.1 Geometric measure of quantum discord

Evaluation of quantum discord, based on the original definition given in [8, 9], involves a difficult optimization procedure and analytical results were obtained only in few cases [25, 26, 27, 28]. To overcome this difficulty Dakic et al introduced a geometric measure of quantum discord [10]. It is defined as the distance between a state  $\rho$  of a bipartite system  $AB$  and the closest classical-quantum state presenting zero discord:

$$D_g(\rho) := \min_{\chi} \|\rho - \chi\|^2 \quad (12)$$

where the minimum is over the set of zero-discord states  $\chi$  and the distance is the square norm in the Hilbert-Schmidt space. It is given by

$$\|\rho - \chi\|^2 := \text{Tr}(\rho - \chi)^2.$$

When the measurement is taken on the subsystem  $A$ , the zero-discord state  $\chi$  can be represented as [9]

$$\chi = \sum_{i=1,2} p_i |\psi_i\rangle\langle\psi_i| \otimes \rho_i$$

where  $p_i$  is a probability distribution,  $\rho_i$  is the marginal density matrix of  $B$  and  $\{|\psi_1\rangle, |\psi_2\rangle\}$  is an arbitrary orthonormal vector set. A general two qubit state writes in Bloch representation as

$$\rho = \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + \sum_i^3 (x_i \sigma_i \otimes \sigma_0 + y_i \sigma_0 \otimes \sigma_i) + \sum_{i,j=1}^3 R_{ij} \sigma_i \otimes \sigma_j \right] \quad (13)$$

where  $x_i = \text{Tr}(\rho(\sigma_i \otimes \sigma_0))$ ,  $y_i = \text{Tr}(\rho(\sigma_0 \otimes \sigma_i))$  are components of local Bloch vectors and  $R_{ij} = \text{Tr}(\rho(\sigma_i \otimes \sigma_j))$  are components of the correlation tensor. The operators  $\sigma_i$  ( $i = 1, 2, 3$ ) stand for the three Pauli matrices and  $\sigma_0$  is the identity matrix. The explicit expression of the geometric measure of quantum discord is given by [10]:

$$D_g(\rho) = \frac{1}{4} (||x||^2 + ||R||^2 - k_{\max}) \quad (14)$$

where  $x = (x_1, x_2, x_3)^T$ ,  $R$  is the matrix with elements  $R_{ij}$ , and  $k_{\max}$  is the largest eigenvalue of matrix defined by

$$K := xx^T + RR^T. \quad (15)$$

Denoting the eigenvalues of the  $3 \times 3$  matrix  $K$  by  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  and considering  $||x||^2 + ||R||^2 = \text{Tr}K$ , we get an alternative compact form of the geometric measure of quantum discord

$$D_g(\rho) = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \quad (16)$$

This form will be more convenient for our purpose.

### 3.2 Geometric measure of quantum discord for pure coherent states

Local unitary operations do not affect the quantum correlations present in a quantum system. In this respect, the geometric quantum discord in the pure state  $|\Omega, \Omega', m, n\rangle$ , partitioned according the scheme (4), can be evaluated by making use of the Schmidt decomposition. Therefore, we write the state  $|\Omega, \Omega', m, n\rangle$  as

$$|\Omega, \Omega', m, n\rangle = \sqrt{\lambda_+} |+\rangle_k \otimes |+\rangle_{n-k} + \sqrt{\lambda_-} |-\rangle_k \otimes |-\rangle_{n-k} \quad (17)$$

where  $|\pm\rangle_k$  denotes the eigenvectors of the reduced density matrix  $\rho_1$  associated with the first subsystem containing  $k$  particles. Similarly,  $|\pm\rangle_{n-k}$  denotes the eigenvectors of the reduced density matrix  $\rho_2$  for the second subsystem. The eigenvalues of the reduced density matrix  $\rho_1$  are given by

$$\lambda_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 - C_{k,n-k}^2} \right)$$

in term of the bipartite concurrence given by

$$C_{k,n-k} = 2|C_{0,0}C_{1,1} - C_{1,0}C_{0,1}| = \frac{\sqrt{1 - p_1^2 p_2^2 \cdots p_k^2} \sqrt{1 - p_{k+1}^2 p_{k+2}^2 \cdots p_n^2}}{1 + p_1 p_2 \cdots p_n \cos m\pi}. \quad (18)$$

Note that the eigenvalues of the reduced matrix density  $\rho_2$  are identical to those of  $\rho_1$ . Separability and maximal entanglement conditions are easily derivable from the equation (18).

Using the prescription discussed in the previous subsection, the derivation of the analytical expression of the geometric quantum discord is very simple. Indeed, one can verify that the matrix  $K$  defined by (15) writes

$$K = \text{diag}(4\lambda_+\lambda_-, 4\lambda_+\lambda_-, 2(\lambda_+^2 + \lambda_-^2)).$$

Thus, using the equation (16), one obtains the bipartite geometric discord present in the state  $|\Omega, \Omega', m, n\rangle$  :

$$D_g(|\Omega, \Omega', m, n\rangle\langle\Omega, \Omega', m, n|) = \frac{1}{2} \frac{(1 - p_1^2 p_2^2 \cdots p_k^2)(1 - p_{k+1}^2 p_{k+2}^2 \cdots p_n^2)}{(1 + p_1 p_2 \cdots p_n \cos m\pi)^2} \quad (19)$$

where we used the mapping defined by the equations (5) and (6) to convert the whole system into a pair of qubit systems. It is remarkable that the result (19) can be rewritten as

$$D_g(|\Omega, \Omega', m, n\rangle\langle\Omega, \Omega', m, n|) = \frac{1}{2} \mathcal{C}_{k,n-k}^2, \quad (20)$$

in term of the bipartite concurrence given by (18). This gives a very simple relation between the entanglement and the geometric quantum discord in a pure multipartite coherent state. It must be noticed that for any bipartite pure state, the quantum discord is exactly the entanglement of formation. It follows that the result (20) establishes a relation between the quantum discord and its geometrized version.

### 3.3 Geometric measure of quantum discord for mixed states

The bipartite mixed density  $\rho_{ij}$  (11), obtained in the second bipartition scheme, can be cast in the following compact form

$$\rho_{ij} = \sum_{\alpha\beta} R_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta \quad (21)$$

where the non vanishing matrix elements  $R_{\alpha\beta}$  ( $\alpha, \beta = 0, 1, 2, 3$ ) are given by

$$R_{00} = 1, \quad R_{11} = 2\mathcal{N}^2 \sqrt{(1 - p_i^2)(1 - p_j^2)}, \quad R_{22} = -2\mathcal{N}^2 \sqrt{(1 - p_i^2)(1 - p_j^2)} q_{ij} \cos m\pi,$$

$$R_{33} = 2\mathcal{N}^2(p_i p_j + q_{ij} \cos m\pi), \quad R_{03} = 2\mathcal{N}^2(p_j + p_i q_{ij} \cos m\pi), \quad R_{30} = 2\mathcal{N}^2(p_i + p_j q_{ij} \cos m\pi).$$

Having mapped the bipartite system  $\rho_{ij}$  into a pair of two qubits (see equations (9) and (10)), now we use it to investigate the pairwise geometric quantum discord according the prescription presented in the subsection 3.1. It is easy, modulo some obvious substitutions, to check that the eigenvalues of the matrix  $K$ , defined in (15), are given by

$$\lambda_1 = 4\mathcal{N}^4 \left[ (1 + p_i^2)(p_j^2 + q_{ij}^2) + 4(p_1 p_2 \cdots p_n) \cos m\pi \right] \quad (22)$$

$$\lambda_2 = 4\mathcal{N}^4 (1 - p_i^2)(1 - p_j^2) \quad (23)$$

$$\lambda_3 = 4\mathcal{N}^4 (1 - p_i^2)(1 - p_j^2) q_{ij}^2 \quad (24)$$



Clearly, we have  $\lambda_3 < \lambda_2$ . Thus, the equation (16) gives

$$D_g = \frac{1}{4} \min\{\lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \quad (25)$$

For mixed states  $\rho_{ij}$ , the explicit expression of geometric quantum discord is

$$D_g = \frac{1}{4} \frac{(1 - p_i^2)(1 - p_j^2)(1 + q_{ij}^2)}{(1 + p_1 p_2 \cdots p_n \cos m\pi)^2} \quad (26)$$

when the condition  $\lambda_1 > \lambda_2$  is satisfied or

$$D_g = \frac{1}{4} \frac{(1 + p_i^2)(p_j^2 + q_{ij}^2) + (1 - p_i^2)(1 - p_j^2)q_{ij}^2 + 4(p_1 p_2 \cdots p_n) \cos m\pi}{(1 + p_1 p_2 \cdots p_n \cos m\pi)^2} \quad (27)$$

in the situation where  $\lambda_1 < \lambda_2$ . It is interesting to note that in the particular case where the system contains only two particles (i.e.,  $n = 2$ ), we have  $q_{12} = 1$  and one can verify that  $\lambda_1 > \lambda_2$  so that the equation (26) reduces to

$$D_g = \frac{1}{2} \frac{(1 - p_1^2)(1 - p_2^2)}{(1 + p_1 p_2 \cos m\pi)^2}$$

which coincides the quantum discord (19) when  $n = 2$  and  $k = 1$ . The results obtained in this section provide us with the explicit expressions of bipartite geometric quantum discord involving nonorthogonal (pure as well as mixed) states. An illustration of these results will be considered in Section 5 for some specific superpositions of multipartite coherent states. But before to do this, we shall, in the following section, discuss the dynamical evolution of pairwise quantum correlations present in the state  $|\Omega, \Omega', m, n\rangle$ .

## 4 Evolution of quantum correlations under dephasing channel

It has been observed that a pair of entangled qubits, interacting with noisy environments, becomes separable in a finite time [29]. The phenomenon of total loss of entanglement, termed in the literature "entanglement sudden death", was experimentally confirmed [30]. The entanglement sudden death depends on the nature of the system-environment interaction. Various decoherence channels, Markovian as well non Markovian, were investigated. In other hand, it has been shown that under some specific channels, when entanglement die in a finite time, the quantum discord vanishes only in asymptotic time [31]. This reflects that the quantum discord is more robust against to decoherence than entanglement.

In this section, we investigate the dynamics of bipartite quantum correlations (entanglement and quantum discord) of the multipartite coherent states  $|\Omega, \Omega', m, n\rangle$  under a dephasing dissipative channel. We use the Kraus operator approach which describes conveniently the dynamics of two qubits interacting independently with individual environments (for more details, see for instance [1]). The time evolution of the bipartite density  $\rho$  can be written compactly as

$$\rho(t) = \sum_{\mu, \nu} E_{\mu, \nu}(t) \rho(0) E_{\mu, \nu}^\dagger(t)$$

where the so-called Kraus operators

$$E_{\mu,\nu}(t) = E_{\mu}(t) \otimes E_{\nu}(t) \quad \text{such that} \quad \sum_{\mu,\nu} E_{\mu,\nu}^{\dagger} E_{\mu,\nu} = \mathbb{I}$$

are the tensorial product of the operators  $E_{\mu}$  describing the one-qubit quantum channel effects. The non-zero Kraus operators for a dephasing channel are given by [31]

$$E_0 = \text{diag}(1, \sqrt{1-\gamma}) \quad E_1 = \text{diag}(0, \sqrt{\gamma}) \quad (28)$$

with  $\gamma = 1 - e^{-\Gamma t}$  and  $\Gamma$  denoting the decay rate.

We first consider the temporal evolution of geometric quantum discord for the pure state given by (4). Using the definition of the dephasing channel the Kraus description (28) and the Schmidt decomposition given by (17), it is simply verified that the matrix  $K$  defined by (15) becomes

$$K(t) = \text{diag}(4e^{-2\Gamma t} \lambda_+ \lambda_-, 4e^{-2\Gamma t} \lambda_+ \lambda_-, 2(\lambda_+^2 + \lambda_-^2))$$

and the geometric discord is then given by

$$D_g(t) = 2 e^{-2\Gamma t} \lambda_+ \lambda_-.$$

Under the dephasing channel, the concurrence evolves as

$$\mathcal{C}_{k,n-k}(t) = e^{-2\Gamma t} \mathcal{C}_{k,n-k}$$

where  $\mathcal{C}_{k,n-k}$  is the concurrence given by (18) and we have

$$D_g(t) = \frac{1}{2} \mathcal{C}_{k,n-k}^2(t).$$

Clearly, for pure states (initially entangled) geometric discord as well as entanglement vanishes only in the asymptotic time limit. This dissipative channel induces an exponential decay of the geometric discord between the two sub-components of the system. This situation becomes completely different for mixed density evolving under a dephasing channel. Indeed, it is easy to check that the density matrix (11) evolves as

$$\rho_{ij}(t) = \mathcal{N}^2 \begin{pmatrix} 2a_i^2 a_j^2 (1+q_{ij} \cos m\pi) & 0 & 0 & 2(1-\gamma) a_i a_j b_i b_j (1+q_{ij} \cos m\pi) \\ 0 & 2a_i^2 b_j^2 (1-q_{ij} \cos m\pi) & 2(1-\gamma) a_i a_j b_i b_j (1-q_{ij} \cos m\pi) & 0 \\ 0 & 2(1-\gamma) a_i a_j b_i b_j (1-q_{ij} \cos m\pi) & 2a_j^2 b_i^2 (1-q_{ij} \cos m\pi) & 0 \\ 2(1-\gamma) a_i a_j b_i b_j (1+q_{ij} \cos m\pi) & 0 & 0 & 2b_i^2 b_j^2 (1+q_{ij} \cos m\pi) \end{pmatrix}, \quad (29)$$

which can be written in the Bloch representation as follows

$$\rho_{ij}(t) = \sum_{\alpha\beta} R_{\alpha\beta}(t) \sigma_{\alpha} \otimes \sigma_{\beta} \quad (30)$$

where all correlation matrix elements are time independent except  $R_{11}(t)$  and  $R_{22}(t)$  which become

$$R_{11}(t) = e^{-\Gamma t} R_{11} \quad R_{22}(t) = e^{-\Gamma t} R_{22}.$$

We will employ the concurrence as a measure of bipartite entanglement for the state (29). We recall that for  $\rho_{AB}$  the density matrix for a pair of qubits  $A$  and  $B$ , the concurrence is [32]

$$C_{AB} = \max \{c_1 - c_2 - c_3 - c_4, 0\} \quad (31)$$

for  $c_1 \geq c_2 \geq c_3 \geq c_4$  the square roots of the eigenvalues of the "spin-flipped" density matrix

$$\varrho_{AB} \equiv \rho_{AB}(\sigma_y \otimes \sigma_y) \rho_{AB}^*(\sigma_y \otimes \sigma_y) \quad (32)$$

where the star stands for complex conjugation and  $\sigma_y \equiv \sigma_2$  is the usual Pauli matrix. Thus, after some algebra, one shows that the entanglement in the state  $\rho_{ij}$  is quantified by the concurrence

$$C_{ij}(t) = 2 \max\{0, \Lambda_1(t), \Lambda_2(t)\}$$

where

$$\begin{aligned} \Lambda_1(t) &= \frac{1}{2} \mathcal{N}^2 \sqrt{(1-p_i^2)(1-p_j^2)} \left[ (1-\gamma)(1+q_{ij} \cos m\pi) - (1-q_{ij} \cos m\pi) \right] \\ \Lambda_2(t) &= \frac{1}{2} \mathcal{N}^2 \sqrt{(1-p_i^2)(1-p_j^2)} \left[ (1-\gamma)(1-q_{ij} \cos m\pi) - (1+q_{ij} \cos m\pi) \right]. \end{aligned}$$

It follows that the concurrence is given by

$$C_{ij}(t) = \frac{1}{2} \frac{\sqrt{(1-p_i^2)(1-p_j^2)}}{1+p_1 p_2 \cdots p_n \cos m\pi} \left[ e^{-\Gamma t} (1+q_{ij}) - (1-q_{ij}) \right]$$

for

$$t < t_0 = \frac{1}{\Gamma} [\ln(1+q_{ij}) - \ln(1-q_{ij})]$$

and the system is entangled. However, for  $t \geq t_0$ , the concurrence is zero and the entanglement disappears, i.e. the system is separable. This results shows that under the dephasing channel, the entanglement suddenly vanishes at  $t = t_0$ . Note that the bipartite system under consideration is in general entangled in the absence of external interaction. Indeed, for  $t = 0$ , the concurrence is given by

$$C_{ij}(0) = \frac{q_{ij} \sqrt{(1-p_i^2)(1-p_j^2)}}{1+p_1 p_2 \cdots p_n \cos m\pi},$$

and vanishes only in the special cases  $q_{ij} = 0$  or  $p_i = 1$  or  $p_j = 1$ . Next, we investigate the temporal evolution of the quantum discord. As above, to study the dephasing channel effect, we compute the evolution of the matrix  $K$  after dephasing interaction. We obtain a diagonal matrix and the corresponding elements are

$$\lambda_1(t) = R_{03}^2 + R_{33}^2 = \lambda_1$$

$$\lambda_2(t) = R_{11}^2(t) = e^{-2\Gamma t} \lambda_2$$

$$\lambda_3(t) = R_{22}^2(t) = e^{-2\Gamma t} \lambda_3$$

where  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  are given by (22), (23) and (24) respectively.

The mixed states  $\rho_{ij}(t)$  are purely classical (zero quantum discord) if and only if  $p_i \rightarrow 0$  for  $i = 1, 2, \dots, n$ . To show this result, we used the criteria provided in [10] to determine states with vanishing quantum discord. This criteria uses the rank of the correlation matrix (the number of non zero eigenvalues) as a simple discord witness and states that if the rank of the correlation tensor is greater than 2 (for qubits), the state has non vanishing quantum discord. It is important to note that in the limiting case  $p_i \rightarrow 0$ , the two states  $|\Omega_i\rangle$  and  $|\Omega'_i\rangle$  approach orthogonality and an orthogonal basis can be constructed such that  $|\mathbf{0}\rangle \equiv |\Omega_i\rangle$  and  $|\mathbf{1}\rangle \equiv |\Omega'_i\rangle$ . In this limit, the state  $|\Omega, \Omega', m, n\rangle$  reduces to the multipartite GHZ state

$$|\text{GHZ}\rangle_n = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \dots \otimes |\mathbf{0}\rangle + e^{im\pi} |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \dots \otimes |\mathbf{1}\rangle). \quad (33)$$

In summary, the dynamic evolutions of geometric quantum discord and entanglement for mixed states, under a dephasing channel, show different behaviors. The life time of pairwise quantum discord, in multipartite coherent states, is infinite while the entanglement suddenly disappears after a finite time of interaction in bipartite mixed states. This agrees well with the results recently obtained by Ferraro et al [33] (see also [34]) showing that the interaction of a quantum system with a dissipative environment can't induce a sudden death of quantum discord contrarily to entanglement which can disappear suddenly and permanently.

## 5 Some special cases

To illustrate the results of the previous sections, we shall focus on the coherent states associated with Weyl-Heisenberg,  $SU(2)$  and  $SU(1, 1)$  symmetries. They are labeled by a single complex variable  $z$ . Also, to simplify further our illustration a special form of (3) is considered. This is given by

$$|z, m, n\rangle = \mathcal{N}(|z\rangle \otimes |z\rangle \otimes \dots \otimes |z\rangle + e^{im\pi} |-z\rangle \otimes |-z\rangle \otimes \dots \otimes |-z\rangle) \quad (34)$$

where here again  $m \in \mathbb{Z}$  and the normalization factor  $\mathcal{N}$  rewrites

$$\mathcal{N} = [2 + 2p^n \cos m\pi]^{-1/2}.$$

The quantity  $p$  is the overlapping  $\langle z | -z \rangle$  between two single particle coherent states of the same amplitude and opposite phase. It is given by

$$\langle z | -z \rangle = \exp(-2|z|^2)$$

for Glauber or Weyl-Heisenberg coherent states ( $z \in \mathbb{C}$ ). For  $su(1, 1)$  algebra, this kernel is given

$$\langle z | -z \rangle = \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^{2k}$$

for group theory or Perelomov coherent states ( $|z| < 1$  and  $k$  is the Chen parameter characterizing the discrete-series representations of  $SU(1, 1)$  Lie group). For  $j$ -spin coherent states or  $su(2)$  coherent states, the quantity  $\langle z | -z \rangle$  is

$$\langle z | -z \rangle = \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^{2j}$$

with  $z \in \mathbb{C}$ . More details concerning coherent states theory can be found in the references [16, 17, 18]. For the pure case obtained in the partitioning scheme defined by (4), the equation (19) gives

$$D_g(|z, m, n\rangle\langle z, m, n|) = \frac{1}{2} \frac{(1 - p^{2k})(1 - p^{2(n-k)})}{(1 + p^n \cos m\pi)^2} \quad (35)$$

In the second partitioning scheme for the state  $|z, m, n\rangle$  given by (11), all the reduced density matrices  $\rho_{ij}$  are identical ( $p_i = p$  for  $i = 1, 2, \dots, n$ ). In this special case, it is simple to see that the eigenvalues of the matrix  $K$  given (22), (23) and (24) take the following forms

$$\lambda_1 = \frac{(p^2 + p^{2(n-2)})(1 + p^2) + 4p^n \cos m\pi}{(1 + p^n \cos m\pi)^2}, \quad (36)$$

$$\lambda_2 = \frac{(1 - p^2)^2}{(1 + p^n \cos m\pi)^2}, \quad (37)$$

$$\lambda_3 = \frac{p^{2(n-2)}(1 - p^2)^2}{(1 + p^n \cos m\pi)^2}, \quad (38)$$

and the geometric quantum discord for the mixed state  $\rho_{12}$  is

$$D_g = \frac{1}{4}(\lambda_2 + \lambda_3) = \frac{1}{4} \frac{(1 + p^{2(n-2)})(1 - p^2)^2}{(1 + p^n \cos m\pi)^2} \quad (39)$$

when

$$(p^2 + 1)(1 + p^{n-2} \cos m\pi) - 2(1 - p^2) \geq 0, \quad (40)$$

or

$$D_g = \frac{1}{4}(\lambda_1 + \lambda_3) = \frac{1}{4} \frac{(p^2 + p^{2(n-2)})(1 + p^2) + 4p^n \cos m\pi + p^{2(n-2)}(1 - p^2)^2}{(1 + p^n \cos m\pi)^2} \quad (41)$$

when

$$(p^2 + 1)(1 + p^{n-2} \cos m\pi) - 2(1 - p^2) \leq 0. \quad (42)$$

The results (39) and (41) follow from the expressions (26) and (27), respectively. The limiting case  $p \rightarrow 1$  must be treated carefully for antisymmetric states ( $m$  odd). In this limit, the eigenvalues of the matrix  $K$  (15) given by (36), (37) and (38) reduce to

$$\lambda_1 = \left(1 - \frac{4}{n}\right)^2 + \left(1 - \frac{2}{n}\right)^2 \quad \lambda_2 = \lambda_3 = \frac{4}{n^2},$$

and the geometric quantum discord is

$$D_g = \frac{2}{n^2}.$$

It is interesting to note that when  $p \rightarrow 1$  ( $z \rightarrow 0$ ), the state (34) with  $m$  odd reduces to a superposition similar to the so-called Werner state [35]. Indeed, when the state  $|z, m = 1, n\rangle$  involves Glauber coherent states, one can verify that

$$|z \rightarrow 0, m = 1, n\rangle \sim |W\rangle_n = \frac{1}{\sqrt{n}}(|1\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes \dots \otimes |0\rangle + \dots + |0\rangle \otimes |0\rangle \otimes \dots \otimes |1\rangle). \quad (43)$$

Here  $|n\rangle$  ( $n = 0, 1$ ) denote the usual harmonic oscillator Fock states. A similar superposition is obtained for the antisymmetric states  $|z, m = 1(\text{mod } 2), n\rangle$  involving  $SU(2)$  and  $SU(1, 1)$  coherent

states when  $z \rightarrow 0$ . This can be done using the contraction procedure to pass from  $SU(2)$  and  $SU(1,1)$  to Weyl-Heisenberg algebra.

In the special case  $n = 2$ , the equations (39), (40), (41) and (42) give

$$D_g = \frac{1}{2} \frac{(1-p^2)^2}{(1+p^2 \cos m\pi)^2}. \quad (44)$$

This expression coincides with the geometric discord given by (35) for  $n = 2$ . Indeed, for  $n = 2$  the density  $\rho_{12}$  is pure. It must be noticed that for  $m$  even, the maximum of the geometric quantum discord is  $1/2$  which is reached in the orthogonal limiting case  $p \rightarrow 0$  (see the figure 1). However, for  $m$  odd, the geometric quantum discord takes the constant value  $1/2$ .

The situation is slightly different for  $n = 3$ . For  $m$  even, the condition (40) (resp. (42)) is satisfied when  $\sqrt{2} - 1 \leq p \leq 1$  (resp.  $0 \leq p \leq \sqrt{2} - 1$ ). It follows that the geometric quantum discord is given by

$$D_g = \frac{1}{4} \frac{p^2(1+p)^2(2+(1-p)^2)}{(1+p^3)^2}$$

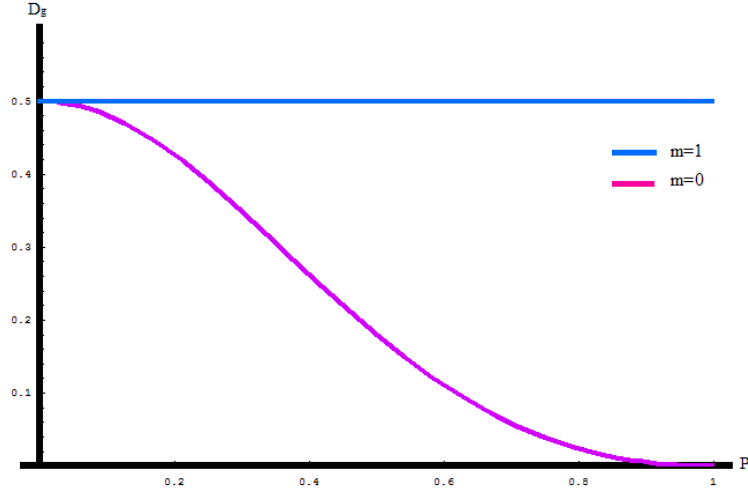
for  $0 \leq p \leq \sqrt{2} - 1$  and

$$D_g = \frac{1}{4} \frac{(1-p^2)^2(1+p^2)}{(1+p^3)^2}$$

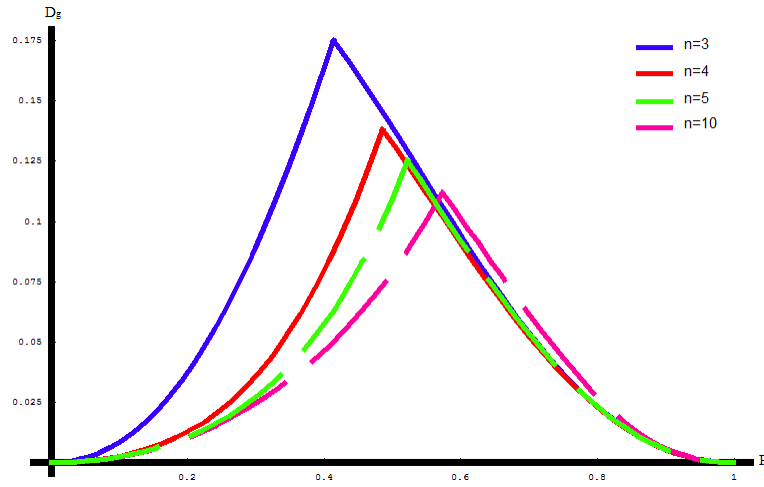
for  $\sqrt{2} - 1 \leq p \leq 1$ . However for antisymmetric states (i.e.  $m$  odd), the condition (42) is satisfied for  $0 \leq p < 1$  and the quantum discord reads as

$$D_g = \frac{1}{4} \frac{p^2(1-p)^2(2+(1+p)^2)}{(1-p^3)^2}.$$

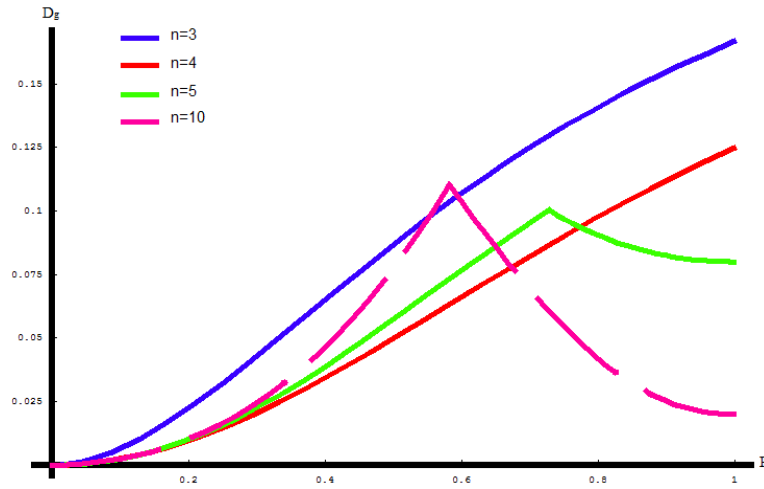
The behavior of geometric quantum discord for mixed states with  $n > 2$  is given in the figures 2 and 3. Figure 2 gives a plot of geometric quantum discord versus the overlapping  $p$  for symmetric multipartite coherent states ( $m$  even). As seen from the figure, after an initial increasing, the quantum discord decreases to vanish when  $p = 1$ . The maximum value of quantum discord occurs when  $\lambda_1$  (36) and  $\lambda_2$  (37) coincide. This maximum decreases as the particle number  $n$  increases. In figure 3, we give a plot of the geometric quantum discord for  $m$  odd (the antisymmetric case) and different values of  $n$ . We have already noticed that in the limit  $p \rightarrow 1$ , we obtain a  $|W\rangle_n$  state (43) and the pairwise quantum discord behaves as  $n^{-2}$  and vanishes only in the limit of large number of particles. Note also that for  $n = 3$  and  $n = 4$ , the geometric quantum discord increases to reach its maximal value in the limit  $p \rightarrow 1$ . However, for  $n \geq 5$ , the maximal value of quantum discord is larger than the one obtained in the limit  $p \rightarrow 1$ . More interestingly, the quantum discord starts increasing to reach its maximal value for some fixed  $p \neq 1$  and decreases after to coincide with the quantum discord characterizing a Werner state obtained for  $p \rightarrow 1$ . It is remarkable that for antisymmetric quantum states containing more than five particles, the maximal value of quantum discord increases as  $n$  increases contrarily to the symmetric states (see figure 2) for which the corresponding maximal value diminishes as  $n$  takes large values.



**Figure 1.** The pairwise geometric quantum discord  $D_g$  versus the overlapping  $p$  for  $n = 2$ .



**Figure 2.** The pairwise geometric quantum discord  $D_g$  versus the overlapping  $p$  for symmetric states.



**Figure 3.** The pairwise geometric quantum discord  $D_g$  versus the overlapping  $p$  for anti-symmetric states

## 6 Summary and concluding remarks

In this paper, we have studied the pairwise quantum correlations in multipartite coherent states. A special attention was paid to bipartite quantum discord. We used the geometric definition of the quantum discord introduced in [10]. We have derived the explicit expressions of this kind of correlation. We also investigated the bipartite entanglement. We used two inequivalent schemes in partitioning the system containing  $n$  modes. The first one consists in splitting the whole system in two parts: one containing  $k$  particles and the second is made of the remaining  $n - k$  particles leading to a pure state system. The second bi-partitioning scheme is realized by tracing out  $n - 2$  particles or modes and gives a mixed state. For each scheme, we mapped the system in a pair of two qubits. This mapping is helpful in investigating the bipartite quantum correlation in a quantum state involving coherent states which constitute the perfect example of non orthogonal states. In particular the obtained quantum correlations (quantum discord as well as entanglement) for antisymmetric superpositions can be viewed as interpolating between ones present in Greenberger-Horne-Zeilinger ( $|\text{GHZ}\rangle_n$ ) and Werner ( $|\text{W}\rangle_n$ ) states. The analytic expressions for geometric quantum discord are corroborated by some numerical analysis. For a pure state (the first partitioning scheme), the geometric quantum discord is proportional to the concurrence. However, for a mixed bipartite state (the second partitioning scheme), the two concepts are completely different. To show that the quantum discord is a kind of correlation beyond the entanglement, we have studied the dynamical evolution of entanglement under a very simple noisy channel (dephasing channel). Indeed, for multipartite coherent states, the pairwise entanglement disappears completely after a finite time interaction while the quantum discord is more resilient. Finally, as prolongation of the present work, it will be interesting to compare the geometric quantum discord and the quantum discord as defined in [8, 9]. Also, the method, discussed in this work to investigate the bipartite correlations, can be extended to examine the multipartite quantum correlations in generalized coherent states in the spirit of the analysis recently proposed in [36].

## References

- [1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).
- [2] V. Vedral, Rev. Mod. Phys. **74** (2002) 197
- [3] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**(2009) 865.
- [4] O. Ghne and G. Tth, Phys. Rep. **474** (2009) 1.
- [5] E. Knill and R. Laflamme, Phys. Rev. Lett. **81** (1998) 5672.
- [6] A. Datta, S.T. Flammia, and C.M. Caves, Phys. Rev. A **72** (2005) 042316; A. Datta and G. Vidal, *ibid.* **75** (2007) 042310; A. Datta, A. Shaji, and C.M. Caves, Phys. Rev. Lett. **100** (2008) 050502.



- [7] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Quantum discord and other measures of quantum correlation, [arXiv:1112.6238](#).
- [8] L. Henderson and V. Vedral, J. Phys. A **34** (2001) 6899; V. Vedral, Phys. Rev. Lett. **90** (2003) 050401; J. Maziero, L.C. Celéri, R.M. Serra, and V. Vedral, Phys. Rev A **80** (2009) 044102.
- [9] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.
- [10] B. Dakic, V. Vedral and C. Brukner, phys. Rev. Lett. **105** (2010) 190502.
- [11] G. Adesso and A. Datta, Phys. Rev. Lett. **105** (2010) 030501; G. Adesso and D. Girolami, Int. J. Quant. Info. **9** (2011) 1773.
- [12] P. Giorda and M.G.A. Paris, Phys. Rev. Lett. **105** (2010) 020503.
- [13] X. Yin, Z. Xi, X-M Lu, Z. Sun and X. Wang, J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 245502.
- [14] B.C. Sanders, Review of Entangled Coherent States, [arXiv:1112.1778](#).
- [15] J.R. Klauder and Bo-S. Skagerstam, Coherent States Applications in Physics and Mathematical Physics, World Scientific, Singapore, 1985.
- [16] A. Perelomov, Generalized Coherent States and Their Applications, Springer, Berlin, 1986.
- [17] S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, Coherent States, Wavelets, and Their Generalizations, Springer, Berlin, 2000.
- [18] J.-P. Gazeau, Coherent States in Quantum Physics, WILEY-VCH Verlag GmbH and Co. KGaA, Weinheim, 2009.
- [19] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J.M. Raimond and S. Haroche, Phys. Rev. Lett. **77** (1996) 4887.
- [20] G.J. Milburn, Phys. Rev. A **33** (1985) 674.
- [21] G.J. Milburn and C.A. Holmes, Phys. Rev. Lett. **56** (1986) 2237.
- [22] B. Yurke and D. Stoler, Phys. Rev. Lett. **57** (1986) 13.
- [23] B. Yurke and D. Stoler, Phys. Rev. A **35** (1987) 4846.
- [24] V. Bužek and P. L. Knight, Progress in Optics **34** (1995) 1.
- [25] S. Luo, Phys. Rev. A **77** (2008) 042303.
- [26] R. Dillenschneider, Phys. Rev. B **78** (2008) 224413.
- [27] M.S. Sarandy, Phys. Rev. A **80** (2009) 022108.

- [28] M. Ali, A.R.P. Rau and G. Alber, Phys. Rev. A **81** (2010) 042105.
- [29] T. Yu and J.H. Eberly, Phys. Rev. Lett. **97** (2006) 140403.
- [30] M.P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S.P. Walborn, P.H. Souto Ribeiro and L. Davidovich, Science **316** (2007) 579.
- [31] T. Werlang, S. Souza, F.F. Fanchini and C.J. Villas Boas, Phys. Rev. A **80** (2009) 024103.
- [32] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78** (1997) 5022.
- [33] A. Ferraro, L. Aolita, D. Cavalcanti, F.M. Cucchietti and A. Acin, Phys. Rev. A **81** (2010) 052318.
- [34] J.H. Cole, J. Phys. A: Math. Theor. **43** (2010) 135301.
- [35] R.F. Werner, Phys. Rev. A **40** (1989) 4277.
- [36] M. Okrasa and Z. Walczak, Eur. Phys. Lett. **96** (2011) 60003.